We studied the following four models:

$\boxed{\text{Model 1}}$ $\frac{dP}{dt} = rP$, $P(t) = P_0 \cdot e^{rt}$
$P(0) = P_0$
However, if $\boxed{P_0 = 0}$ then $\boxed{P(t) = 0}$ for all $t > 0$.

$\boxed{\text{Model 2}}$ $\frac{dP}{dt} = rP(1 - \frac{P}{K})$

$$P(t) = \frac{K}{1 + (\frac{K}{P_0} - 1)e^{-rt}}$$

If $\boxed{P_0 = 0}$ then $\boxed{P(t) = 0}$ for all $t > 0$
and if $\boxed{P_0 = K}$, then $\boxed{P(t) = K}$ for all $t > 0$

$\boxed{\text{Model 3}}$ $\frac{dP}{dt} = rP(1 - \frac{P}{K})(P - a)$

If $P_0 = 0 \Rightarrow P(t) = 0$ for any t.
If $P_0 = a \Rightarrow P(t) = a$ for any t
If $P_0 = K \Rightarrow P(t) = K$ for all t.

$\boxed{\text{Model 4}}$ $\frac{dH}{dt} = \alpha(H - A) < 0$, $H > A$, $\alpha < 0$

if $H_0 = A \Rightarrow H(t) = A$ for all t.

For example, we see from Model 2 if $P = K$ or $P = 0$ then $\frac{dP}{dt} = 0$ ~~then~~ implying that $P(t)$ is constant.

Constant solutions form a very special class of solutions of autonomous dif. eq. They are called <u>equilibria</u>

$\boxed{\text{Definition}}$ Consider an autonomous dif. eq-n of the form $\frac{dy}{dx} = f(y)$ (where we think of x as time). If y^* satisfies $f(y^*) = 0$ then y^* is <u>an equilibrium</u> of $\frac{dy}{dx} = f(y)$

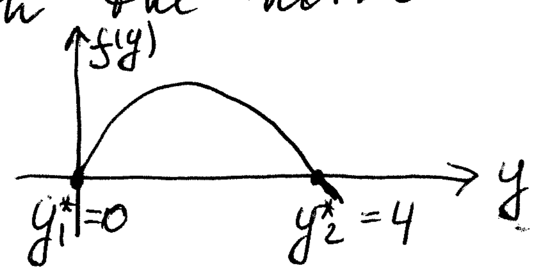
$\boxed{\text{Example}}$ $\frac{dy}{dx} = y(4-y) = f(y)$

Find the equilibria. To find the equilibria, we set $f(y) = 0$ or $\frac{dy}{dx} = 0$ which yields

$(4-y) \cdot y = f(y) = 0$

$y_1^* = 4$; $y_2^* = 0$

Graphically, this means that if we graph $f(y)$ as a function of y, then the equilibria are points of intersection of $f(y)$ with the horizontal axis (the y-axis).



Since we are discussing autonomous dif. eq; the derivative ($\frac{dy}{dx}$) is a function of y only! This allows us to graph $\frac{dy}{dx}$ as a function of y or to graph $f(y)$ as a function of y (since $\frac{dy}{dx} = f(y)$). Then we can use the graph of $f(y)$ in the $(y, f(y))$ plane to say something about $\frac{dy}{dx}$. Namely, if the current value $\bar{y}$ is such that $f(\bar{y}) > 0$ then $\left.\frac{dy}{dx}\right|_{y=\bar{y}} > 0$ and y will increase at $\bar{y}$ as a function of x. If y is such that $f(y) < 0$ (that is $\frac{dy}{dx} < 0$), then y will decrease as a function of x

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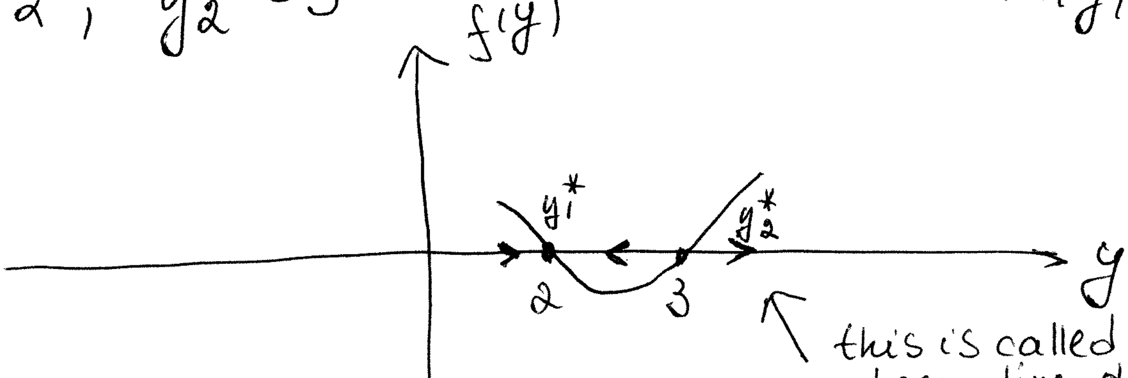
$\boxed{\text{Example}}$ $\frac{dy}{dx} = (2-y)(3-y) = f(y)$

Equilibria: $\frac{dy}{dx} \equiv 0$ or $f(y) = 0$

$y_1^* = 2$, $y_2^* = 3$

the $(y, f(y))$ plane



this is called a <u>phase-line diagram</u>.

Pick $\bar{y}_1 \in (-\infty, 2)$

for example, $\bar{y}_1 = 1$ $\quad f(\bar{y}_1) = f(1) = (2-1)(3-1) = 2 > 0$

$\left.\frac{dy}{dx}\right|_{\bar{y}_1} > 0$ $\quad y(x)$ increases as x.

(or any other point from the interval)

Pick $\bar{y}_2 \in (2, 3)$

$\bar{y}_2 = \frac{5}{2}$.

$f(\bar{y}_2) = f\left(\frac{5}{2}\right) = \left(2 - \frac{5}{2}\right)\left(3 - \frac{5}{2}\right) < 0$

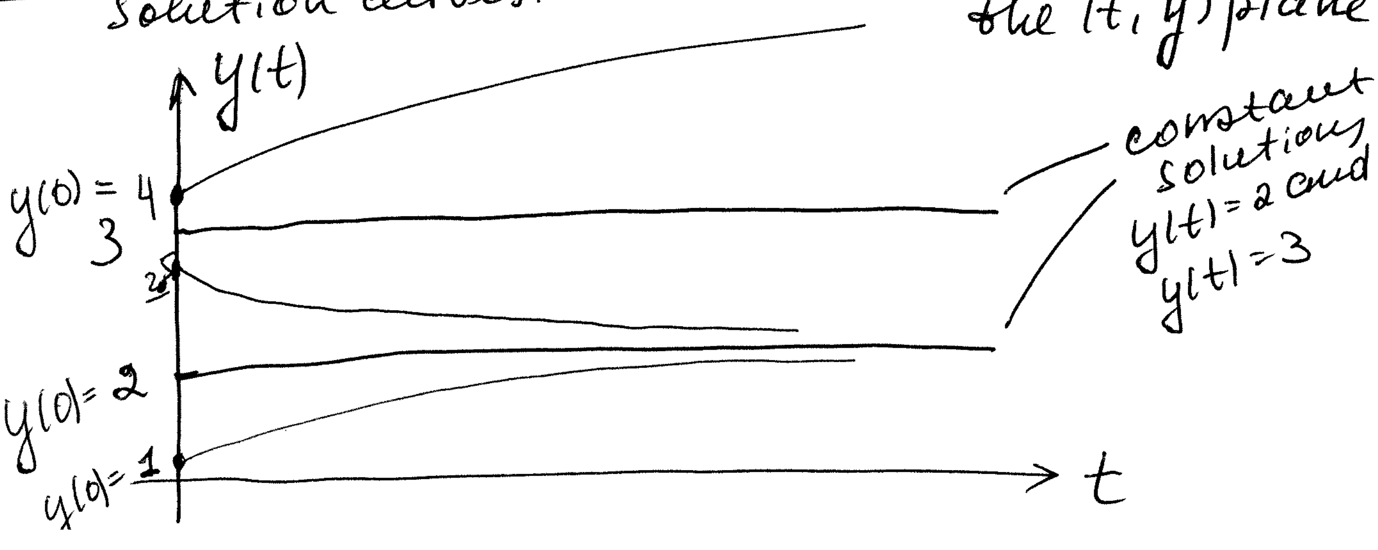
$\left.\frac{dy}{dx}\right|_{\bar{y}_2} < 0 \Rightarrow y(x)$ decreases as function of x for any point from $(2, 3)$

→ $\bar{y}_3 = 4$

$f(\bar{y}_3) = f(4) = (2-4)(3-4) > 0$.

$\left.\frac{dy}{dx}\right|_{\bar{y}_3} > 0 \Rightarrow y(x)$ increases as x for any point from $(3, +\infty)$.

Solution curves: $\quad$ the (t, y) plane

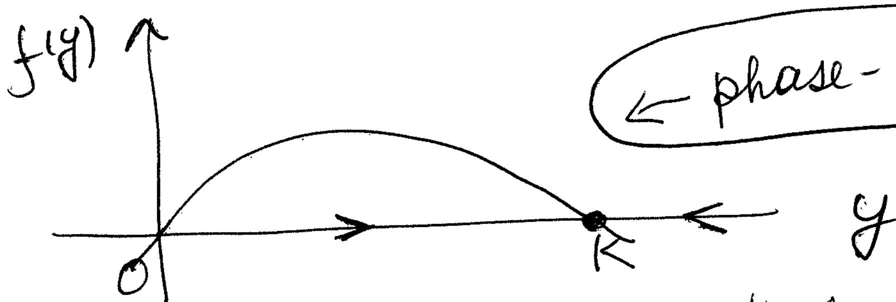


Thus, without solving the equation, we were able to graph the solution curves to the equation. The only information that we used is information about equilibria and their stability.

$\boxed{\text{Model 2}}$ $\quad \frac{dP}{dt} = rP\left(1-\frac{P}{K}\right) = f(P)$

$$\left.\begin{array}{l} P_1^* = 0 \\ P_2^* = K \end{array}\right\} \text{equilibria}$$

the $(y, f(y))$ plane



← phase-line diagram.

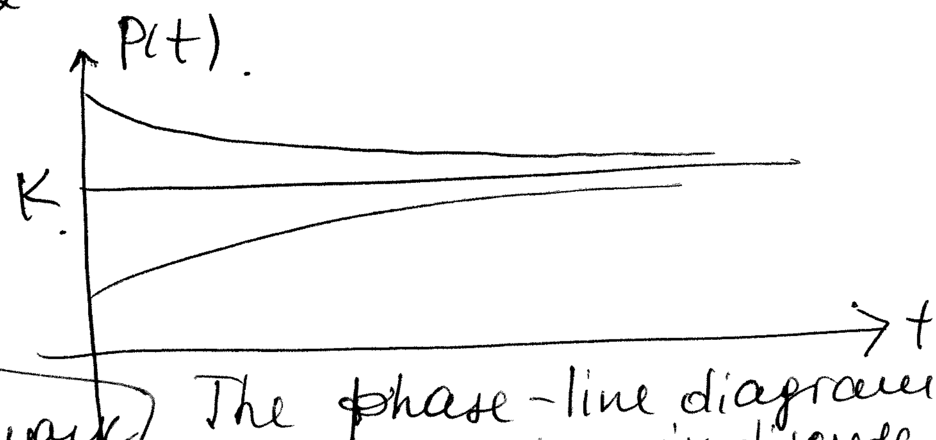
time is given implicitly in phase-line diagrams.

$\bar{P} = \frac{K}{2}$ $\quad \left.\frac{dP}{dt}\right|_{\bar{P}=\frac{K}{2}} = r\cdot\frac{K}{2}\left(1-\frac{1}{2}\right) > 0.$

$P_1^* = 0$ is unstable equilibrium

$P_2^* = K$ is stable eq-m.

the $(t, P(t))$ plane



$\boxed{\text{Remark}}$ The phase-line diagram is analogue of cobwebbing in discrete time dyn. systems.

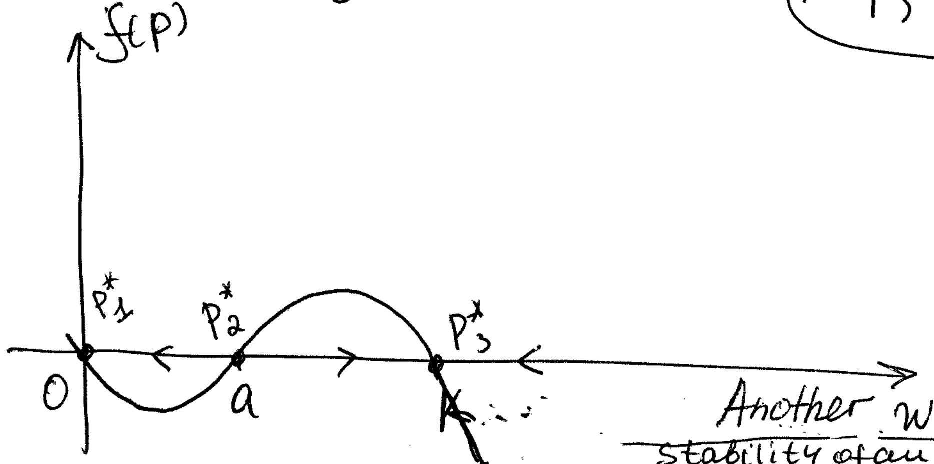
$$\frac{dP}{dt} = rP(P-a)\left(1-\frac{P}{K}\right) = f(P)$$

$f(P)=0 \qquad P_1^* = 0$

$\qquad\qquad\quad P_2^* = a$

$\qquad\qquad\quad P_3^* = K.$

the $(P, f(P))$ plane



Another way to determine stability of an equilibrium.

$\boxed{\text{Stability Criterion}}$: Consider the dif. eq-n $\frac{dy}{dt} = f(y)$, where $f(y)$ is a differentiable f-n. Assume that y^* is an equilibrium, that is $f(y^*)=0$. Then

y^* is stable if $f'(y^*) < 0$ and

y^* is unstable if $f'(y^*) > 0$.

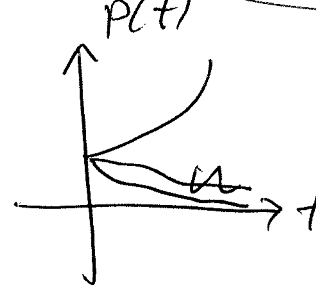
Exponential model: (Model 1.)

$(t, P(t))$ plane

$$\frac{dP}{dt} = rP = f(P)$$

$f'(P) = r$ if $r > 0$ $P^* = 0$ is unstable.

$\qquad\qquad$ if $r < 0$ $P^* = 0$ is stable.



Logistic eq-n (Model 2)

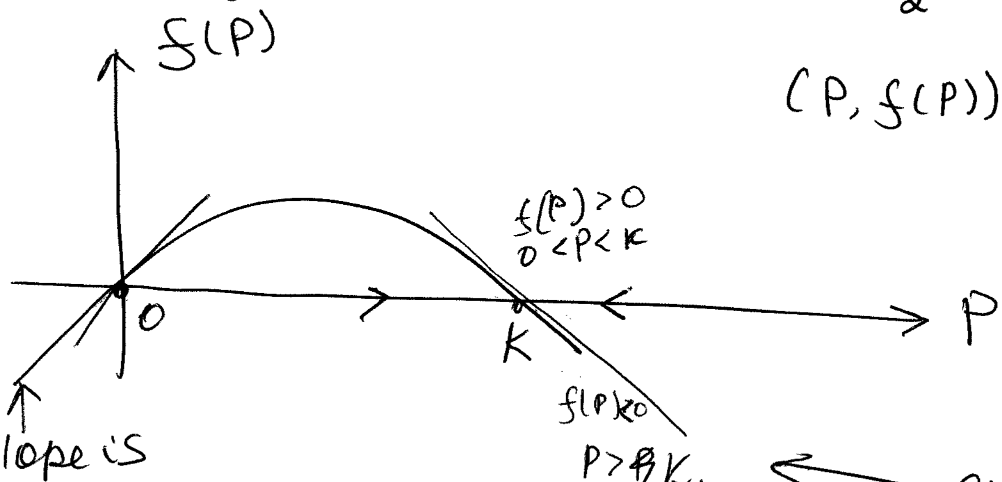
$$\frac{dP}{dt} = rP\left(1-\frac{P}{K}\right) = f(P)$$

$$f'(P) = r\left(1-\frac{P}{K}\right) + rP\cdot\left(-\frac{1}{K}\right) = r - \frac{2rP}{K}.$$

$f'(P_1^*) = f'(0) = r$ if $r>0$ $\quad P_1^* = 0$ is unstable equilibrium.

$f'(P_2^*) = f'(K) = r - 2r = -r < 0$ $\quad P_2^* = K$ is stable equilibrium

(P, f(P)) plane



Slope is positive at the first eq-m.

Slope of f(P) is negative at the second eq-m.

Allee Effect (Model 3)

$$\frac{dP}{dt} = rP(P-a)\left(1-\frac{P}{K}\right) = f(P)$$

$$f'(P) = \left[rP(P-a)\right]'\left(1-\frac{P}{K}\right) + rP(P-a)\cdot\left(-\frac{1}{K}\right) =$$

$$= \cancel{r\frac{2rP}{K}}\left[r(P-a)+rP\right]\left(1-\frac{P}{K}\right) - \frac{rP}{K}(P-a) =$$

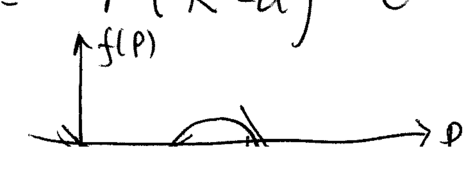
$$= (2rP - ra)\left(1-\frac{P}{K}\right) - \frac{rP}{K}(P-a)$$

$f'(P_1^*) = f'(0) = -ra < 0 \Rightarrow P_1^* = 0$ is stable

$f'(P_2^*) = f'(a) = ra\left(1-\frac{a}{K}\right) > 0 \Rightarrow P_2^* = a$ is unstable

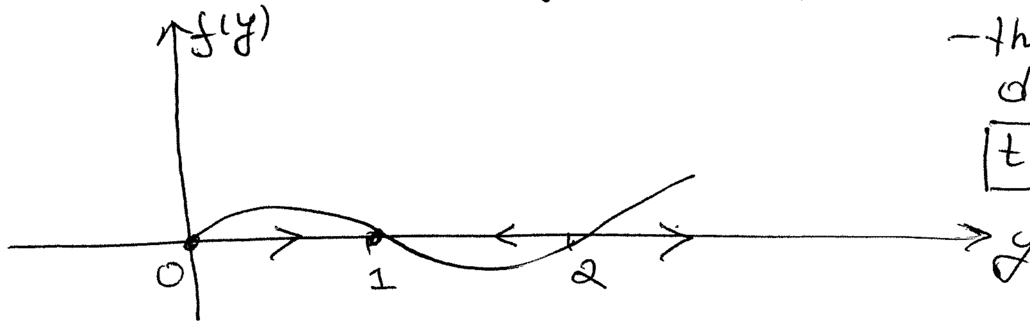
$f'(P_3^*) = f'(K) = -r(K-a) < 0 \Rightarrow P_3^* = K$ is stable.

(P, f(P)) - plane.



Example $\quad \dfrac{dy}{dx} = y(y-1)(y-2) = f(y)$

$f(y) = 0 \Rightarrow y_1^* = 0, \; y_2^* = 1, \; y_3^* = 2$



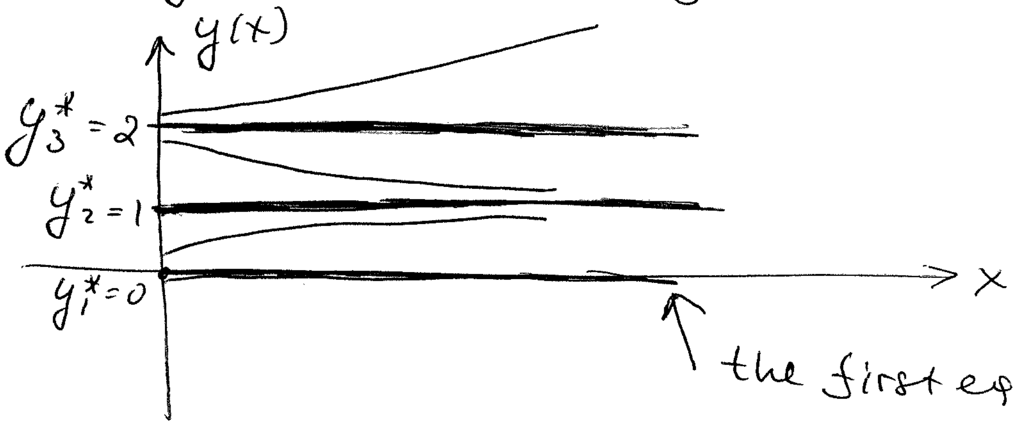
— the phase-line diagram.

the $(y, f(y))$ plane

take $\bar{y} = 0.5 \in (0,1)$ $\quad f(0.5) > 0$

$\bar{y} = 1.5 \in (1,2)$ $\quad f(1.5) < 0$

$\bar{y} = 3$ $\quad f(3) > 0.$



the solution curve.

the (x, y) plane

the first equilibrium $y_1^* = 0$

Stability Criterion

$f'(y) = (y-1)(y-2) + y(y-2) + y(y-1)$

$f'(0) = (-1)(-2) = 2 > 0 \Rightarrow y_1^* = 0$ is unstable

$f'(1) = 1 \cdot (-1) = -1 < 0 \Rightarrow y_2^* = 1$ is stable

$f'(2) = 2 \cdot (2-1) = 2 > 0 \Rightarrow y_3^* = 2$ is unstable.

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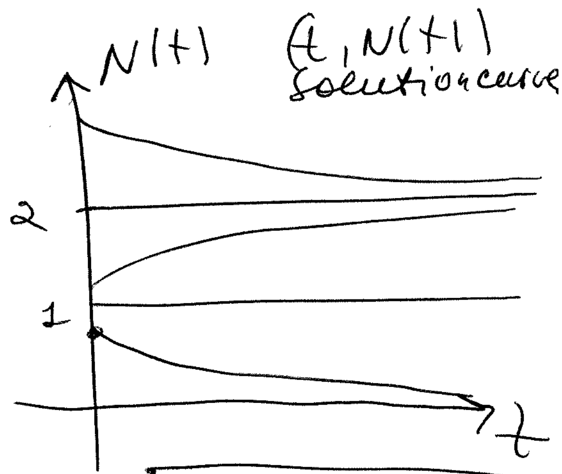
Suppose the population size of some species follows the model:

$$\frac{dN}{dt} = \frac{3N^2}{2+N^2} - N = f(N)$$



$f(N)=0 \qquad \dfrac{3N^2}{2+N^2} - N = 0$

$$N\left(\frac{3N}{2+N^2} - 1\right) = 0$$

$N_1^* = 0 \qquad \dfrac{3N}{2+N^2} = 1$

$N^2 - 3N + 2 = 0$

$N_2^* = 1$

$N_3^* = 2.$

$N(0.5)$ - ?
$N(1.5)$ - ?
$N(6)$ - ?
!

Separation of variables ?

The best method is Stability Criterion

$$f'(N) = \left(\frac{3N^2}{2+N^2}\right)' - 1 = \frac{6N(2+N^2) - 3N^2 \cdot 2N}{(2+N^2)^2} - 1 =$$

$$= \frac{12N}{(2+N^2)^2} - 1$$

$f'(0) = -1 \Rightarrow N_1^* = 0$ is stable

$f'(1) = \dfrac{12}{9} - 1 > 0 \Rightarrow N_2^* = 1$ is unstable

$f'(2) = \dfrac{24}{36} - 1 < 0 \Rightarrow N_3^* = 2$ is stable.